# Algorithms for Buffering Fuzzy Maps

**Hans W. Guesgen**
Computer Science Department
University of Auckland
Private Bag 92019, Auckland, New Zealand
hans@cs.auckland.ac.nz

**Joachim Hertzberg**
GMD, AiS.ARC
Schloss Birlinghoven
53754 Sankt Augustin, Germany
joachim.hertzberg@gmd.de

**Abstract**

In this paper, we show how standard GIS operations like the complement, union, intersection, and buffering of maps can be made more flexible by using fuzzy set theory. In particular, we present a variety of algorithms for operations on fuzzy maps, focusing on buffer operations for fuzzy maps.

## Introduction

Although geographic information systems have been used for quite a while (Coppock & Rhind 1991), their functionality has changed only little over the years. In spite of their name, geographic information systems have so far been mostly geometric in nature, ignoring the temporal, thematic, and qualitative dimensions of geographic features (Molenaar 1996; Sinton 1978; Usery 1996). There are numerous attempts to overcome these limitations. For example, a variety of papers (Frank 1992; Goodchild 1992; Gupta, Weymouth, & Jain 1991; Herring 1991; 1992; Raper & Maguire 1992) deal with extensions of the data model, while Allen's work and its derivatives (Allen 1983; Freksa 1990; Guesgen 1989; Hernández 1991; Mukerjee & Joe 1990) form the basis for numerous temporal and qualitative endeavors to extend geographic information systems (Egenhofer & Golledge 1997; Frank 1994; 1996; Peuquet 1994). Applications of fuzzy techniques are most commonly found in remote sensing literature but (Altmann 1994; Brimicombe 1997; Molenaar 1996; Plewe 1997) provide examples that the inherent fuzziness of geographic features becomes increasingly acknowledged in geographic information science as well.

In many geographic information systems, the extraction of new information from stored spatial data is achieved through map overlap. New maps are computed from existing ones by applying one of the following operations:

- Buffer operations, which increases the size of an object by extending its boundary.
- Set operations, such as complement, union, and intersection.

These operations are exact quantitative operations. Humans, on the other hand, often prefer a vague, uncertain, or qualitative operation over an exact quantitative one. For example, instead of requesting all locations on a map that are at most 2810m away from the sea, it would be more adequate from the cognitive viewpoint to request all location that are close to the sea (Clementini, Di Felice, & Hernández 1997). This, however, requires some kind of vague, uncertain, or qualitative buffer operation. We have introduced such an operation, together with other similar operations, in previous papers (Guesgen & Albrecht 1998; Guesgen & Histed 1996) by using fuzzy set theory, but have not discussed efficient algorithms for the fuzzy operations.



## Fuzzifying Maps

In the following, we restrict ourselves to raster-based maps. Such a map consists of a grid of cells whose values specify certain attributes of the locations represented by the map. In the simplest case, the cell values are restricted to 0 and 1, where 0 signals the absence and 1 the presence of a certain attribute, like the attribute of a location being part of a road, waterway, residential area, commercial area, rural area, etc.

In some cases, there is a crisp boundary between locations that have a certain attribute and those that don't have that attribute, but often this is not the case. For example, it is not always clear where a rural area stops and a residential areas start, or where a forest is not a forest any more. To cater for this fact, we extend the range of cell values from the set $\{0, 1\}$ to the interval $[0, 1]$, and thereby convert a regular raster map into a fuzzy raster map. Given a cell $x$ in the fuzzy raster map, $\mu(x) \in [0, 1]$ indicates the degree to which $x$ has the attribute represented by the map. The function $\mu(x)$ is called the membership function of the fuzzy raster map.

Performing a set operation (complement, union, and intersection) on fuzzy raster maps is straightforward. There are several ways of defining the complement, union, and intersection of membership functions (Driankov, Hellendoorn, & Reinfrank 1996), but they all have in common that they are defined cell-wise for all cells $L$ in the fuzzy raster map $L$. In the case of the original max/min scheme (Zadeh 1965), the membership functions for the complement, union, and intersection are defined as follows, where $\mu_1$ and $\mu_2$ denote the membership functions of the underlying maps and $\mu_3$ the one of the resulting map:

| | |
|---|---|
| Complement: | $\forall x \in L : \mu_3(x) = 1 - \mu_1(x)$ |
| Union: | $\forall x \in L : \mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$ |
| Intersection: | $\forall x \in L : \mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$ |

Since the membership functions for the complement, union, and intersection are defined cell-wise, an algorithm for performing a set operation on fuzzy raster maps can just iterate through the set of cells and compute a new value for each cell based on the given value(s) for that cell, which means the algorithm is linear in the number of cells, i.e, its complexity is $O(|L|)$.

## Buffer Operations

Unlike the set operations, buffer operations cannot be defined cell-wise. They usually involve a number of cells that are in the same neighborhood. If any of these has a value of 1, then the value of $x$ is changed to 1; otherwise it remains unchanged. In other words, we compute the maximum of the value of $x$ and the values of all cells in the neighborhood of $x$. A fuzzy raster map can be buffered in a similar way, resulting in values from the interval $[0, 1]$ rather than the set $\{0, 1\}$.

Although buffering a fuzzy raster map as indicated above might be of use for many applications, we do not want to restrict ourselves to crisp buffer operations for fuzzy raster maps. Rather, we want the buffer operation to depend on the proximity of the cells under consideration. For example, if there is an area on the map with very high membership grades, then the buffer operation should assign high membership grades to cells that are very close to that area, medium high membership grades to cells close to the areas, and low membership grades to cells further away.

One way to achieve this behavior is to determine the direct neighbors of a cell and to apply a buffer function to determine the new membership grade of these neighbors. There are two types of direct neighbors:

- Edge-adjacent (4-adjacent) neighbors, or edge neighbors for short. Two cells of the grid are edge neighbors, if and only if they have an edge in common.
- Vertex-adjacent (8-adjacent) neighbors, or vertex neighbors for short. Two cells of the grid are vertex neighbors, if and only if they have a vertex in common.

A buffer function is a monotonically increasing function $\beta : [0, 1] \rightarrow [0, 1]$ that satisfies the following condition:

$$\forall m \in [0, 1] : \beta(m) \leq m$$

If $x_0$ is a neighbor of $x_1$, then the new membership grade of $x_1$ is determined by the maximum of the old membership grade of $x_1$ and the value of the buffer function applied to the membership grade of $x_0$:

$$\mu(x_1) \leftarrow \max\{\mu(x_1), \beta(\mu(x_0))\}$$

Since updating the membership grade of $x_1$ can have an impact on the membership grades of the neighbors of $x_1$, the update process has to be repeated for all cells of the map over and over again until a stable situation is obtained.

Figure 1 shows an illustration of a partially buffered map. We assume that in this example the original map had only membership grades of 0 (unfilled white areas) and 1 (striped dark grey areas). The buffer operation uses the vertex neighbor relation to increase the membership grades of an unfilled

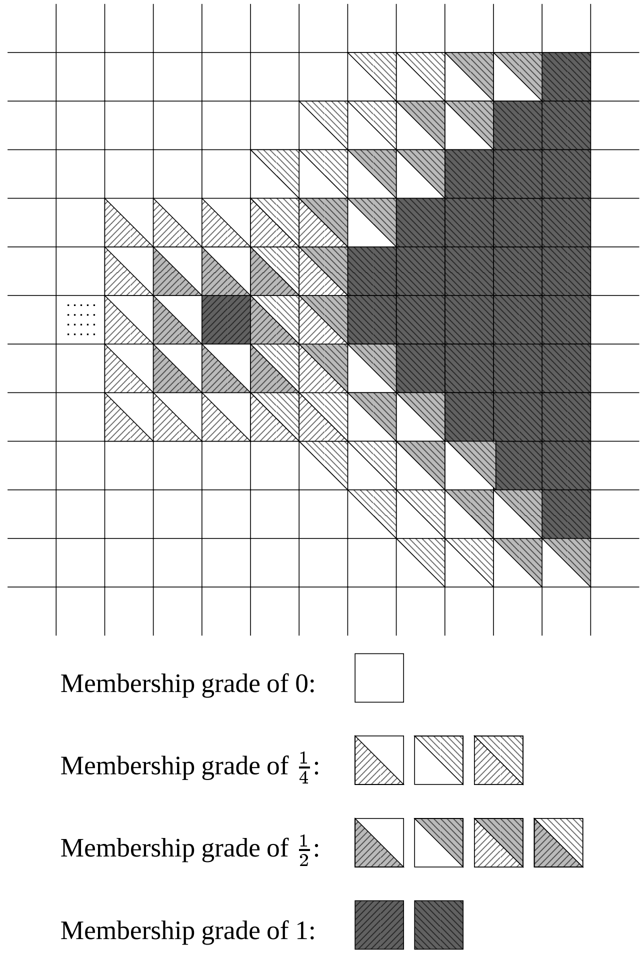


Figure 1: A partially buffered map illustrated by using a grey scale to indicate different membership grades. The upper right part of each cell indicates the value derived from the striped dark grey cells on the right, whereas the lower left part of the cell indicates the value derived from the solitary dark grey cell. The overall value of the cell is the maximum of the two values.

cell to a value of $\frac{1}{2}$ (striped light grey areas) if there is at least one neighbor with a value of 1, or to a value of $\frac{1}{4}$ (striped white areas) if there is at least one neighbor with a value of $\frac{1}{2}$ but no neighbors with a value of 1.

## Algorithms for Buffering

A brute-force algorithm for buffering a fuzzy raster map is shown in Figure 2. The algorithm visits each cell of the map and updates its membership grade based on the membership grades of the neighboring cells. If any of the membership grades is changed, the algorithm repeats the updating process until all membership grades become stable. More precisely, the algorithm applies the buffer function $\beta$ to the membership grade $\mu(x_0)$ of a cell $x_0$ and uses the result to update the membership grades of the edge neighbors of $x_0$ ($k = 4$) or the vertex neighbors of $x_0$ ($k = 8$), respectively.

**Brute-Force Buffering**

Let $\mu$ be the membership function of the map.

Let $\beta$ be a buffer function.

Let $L$ be the set of all cells in the map to be buffered.

Repeat until $\mu$ is stable:

  For each $x_0 \in L$ do:

    For all neighbors $x_i$ of $x_0$ do:

      $\mu(x_i) \leftarrow \max\{\mu(x_i), \beta(\mu(x_0))\}$

Figure 2: A brute-force algorithm for buffering raster fuzzy maps.

**Buffering by Local Propagation**

Let $\mu$, $\beta$, and $L$ be defined as before (Figure 2).

While $L \neq \emptyset$ do:

  Select $x_0 \in L$.

  $L \leftarrow L - \{x_0\}$

  For all neighbors $x_i$ of $x_0$ do:

    $\mu(x_i) \leftarrow \max\{\mu(x_i), \beta(\mu(x_0))\}$

    If $\mu(x_i)$ has changed, then $L \leftarrow L \cup \{x_i\}$

Figure 3: A local propagation algorithm for buffering fuzzy raster maps.

Note that membership grades of the original map are lower bounds for the membership grades of the new map.

Since the algorithm revisits each cell when repeating the updating process, even the ones whose neighbors have not been changed in the previous iteration, it performs many unnecessary checks. An improved approach is to keep track of the changed cells and to revisit a cell only if at least one of its neighbors has been changed. The algorithm in Figure 3 achieves this by applying the principle of local propagation: the membership grade of a cell is propagated to the neighbors of the cell, which are then put on to the list of cells to be visited in the future.

The local propagation algorithm is guaranteed to terminate. To see this, note that maps have finite sets of cells $L$, and hence a finite number of membership grades. The transitive closure of the neighborhood relation is also finite, which means that the buffer operation is applied a finite number of times to a finite number of values. Therefore, the number of new membership grades is limited and with that the number of possible changes. Since cells are only put back into the set $L$ if the membership grade has changed, $L$ must eventually become empty.

Although the propagation algorithm is guaranteed to terminate, it may take large number of cells to be revisisted before $L$ before finally becomes empty, the reason being that it is always possible for a cell to receive a larger membership grade because of the buffer operation. To prevent this

**Buffering with Ordered Cells**

Let $\mu$, $\beta$, and $L$ be defined as before (Figure 2).

While $L \neq \emptyset$ do:

  Select $x_0 \in L$ such that $\mu(x_0)$ is maximal in $L$.

  $L \leftarrow L - \{x_0\}$

  For all neighbors $x_i$ of $x_0$ do:

    $\mu(x_i) \leftarrow \max\{\mu(x_i), \beta(\mu(x_0))\}$

Figure 4: An algorithm for buffering fuzzy maps using ordered cells.

from happening, we can select a cell from $L$ with a maximum membership grade. The grade for such a cell cannot be increased by any buffer operation $\beta(x)$, since $\beta(x) \leq x$ for all $x \in [0, 1]$, and therefore buffering the neighbors of a cell with maximum membership grade results in assigning a final membership grade to the neighbors of that cell. This means that none of the cell have to be revisited. The improved algorithm is shown in Figure 4.

## From Iterative Buffering to Global Buffering

Although propagating the result of a buffer function $\beta$ locally through a fuzzy raster map is a reasonable way to buffer such a map, it does not cater for global effects, as the membership grade of a cell is determined by its original membership grade and the grade of its immediate neighbors, but not by the membership grade of cells further away from the cell under consideration. To achieve a more global effect, we replace $\beta$ with a global buffer (or proximity) function $\psi$ that is applied not only to the membership grades of the neighbors of a given cell $x_0$ but potentially to any cell $x$ in the map. The function $\psi$ has two arguments, one of which is $\mu(x_0)$, the membership grade of $x_0$, and the other is $\delta(x, x_0)$, the distance between $x$ and $x_0$, which can be defined as follows:

1. $\delta(x_0, x_0) = 0$
2. $\forall x \neq x_0$ :
   $\delta(x, x_0) = \min\{\delta(x', x_0) \mid x' \text{ neighbor of } x)\} + 1$

We require $\psi$ to be monotonically increasing in the first argument, i.e., the larger the membership grade of $x_0$, the larger the value of $\psi$, and monotonically decreasing in the second argument, i.e., the further away $x$ is from $x_0$, the smaller the value of $\psi$. We further require that the value of $\psi$ never exceeds the value of the first argument:

$$\forall m \in [0, 1] \text{ and } \forall d \in [0, \infty) : \psi(m, d) \leq m \qquad (1)$$

The update of a membership grade is computed in a similar way as before:

$$\mu(x) \leftarrow \max\{\mu(x), \psi(\mu(x_0), \delta(x, x_0))\}$$

In addition to that, we have to ensure that the resulting membership grades are plausible from the intuitive point of view. In particular, we want to avoid that a local effect overrides a more global one if they originate in the same cell. For

**Global Brute-Force Buffering**

Let $\mu$ be the membership function of the map.

Let $\psi$ be a global buffer function.

Let $L$ be the set of all cells in the map to be buffered.

For each $x_0 \in L$ do:

  For all $x \in L - \{x_0\}$ do:

   $\mu(x) \leftarrow \max\{\mu(x), \psi(\mu(x_0), \delta(x, x_0))\}$

Figure 5: A brute-force algorithm for buffering fuzzy raster maps using a global buffer function.

example, if a cell $x_0$ has a distance of 1 to a cell $x_1$ and a distance of 2 to a cell $x_2$, then $\psi(\psi(\mu(x_2), 1), 1)$ should not exceed $\psi(\mu(x_2), 2)$, i.e., the new membership grade of $x_0$ is influenced by the membership grade of $x_2$ directly rather than the propagation of that membership from $x_2$ through $x_1$ to $x_0$. We can enforce this property by requiring the following:

$$\forall m \in [0,1] \text{ and } \forall d_0, d_1, d_2 \in [0, \infty): \qquad (2)$$
$$d_2 = d_1 + d_0 \Longrightarrow \psi(m, d_2) \geq \psi(\psi(m, d_1), d_0)$$

The function $\psi(m, d) = \frac{m}{1+d}$, for example, satisfies this criterion, whereas $\psi(m, d) = \frac{m}{1+d^2}$ does not.

If we require equality instead of inequality in Formula (2), we achieve the same effect as with the function $\beta$ as introduced in one of the previous sections. If $\psi(m, d_2) = \psi(\psi(m, d_1), d_0)$, then the new membership grade of a cell $x$ with distance $d$ from cell $x_0$ can be computed by applying $\psi$ successively to the membership grade of $x_0$, i.e., by defining $\beta(m) \equiv \psi(m, 1)$:

$$\mu(x) \leftarrow \max\{\mu(x), \underbrace{\psi(\psi(\ldots\psi}_{d}(\mu(x_0), 1)\ldots))\}$$

A brute-force algorithm for buffering a fuzzy map using a global buffer function rather than a local one can be obtained by extending the update operations in the algorithm of Figure 2 to all cells in the map. The resulting algorithm is shown in Figure 5. The algorithm repeatedly iterates through the set of cells, using the membership grades of a cell to update the membership grades of the other cells. This is done regardless of whether the membership grade of a cell can possibly have an effect on other cells or not. An improvement can be achieved by using only those cells that have the potential to influence other cells. This is the case if the current membership grade of the cell is not minimal and was not derived from the membership grade of another cell through buffering. Cells with minimal membership grade cannot increase the membership grade of another cell during buffering, because the buffer operation always returns a value smaller than or equal to the membership grade of the cell that is used as argument of the buffer operation (cf. Formula (1)). A cell whose membership grade was derived from the membership grade of another cell through buffering cannot make any contribution because the other cell has spread its influence to all cells of the map already, and since global effects dominate local ones (cf. Formula (2)), the current membership grade of the cell under consideration does not have any additional effect.

**Global Buffering with Ordered Cells and Cutoffs**

Let $\mu$, $\psi$, and $L$ be defined as before (Figure 5).

$L' \leftarrow L - \{x \mid \mu(x) \text{ is minimal in } L\}$

While $L' \neq \emptyset$ do:

  Select $x_0 \in L'$ such that $\mu(x_0)$ is maximal in $L'$.

  $L' \leftarrow L' - \{x_0\}$

  For all $x \in L - \{x_0\}$ do:

   $\mu(x) \leftarrow \max\{\mu(x), \psi(\mu(x_0), \delta(x, x_0))\}$

   If $\mu(x)$ has changed, then $L' \leftarrow L' - \{x\}$

Figure 6: An algorithm for buffering fuzzy raster maps using a global buffer function, ordered cells, and cutoffs.

Figure 6 shows an improved algorithm, which restricts the outer loop to the set of cells that might have an influence on other cells. Initially, this set contains all cells of the map. However, when a cell is detected whose membership grade is updated through a buffer operation, the cell that was updated is removed from the set of influential cells, because it won't have any effect on the membership grades of other cells in a future iteration. In addition to that, the cells to be buffered are selected according to their membership grades. Cells with large membership grades are more likely to cause a cutoff than those with smaller grades. It therefore makes sense to consider cells with large membership grades first.

## Conclusion

In the first part of the paper, we introduced algorithms for buffering fuzzy raster maps with a local buffer function. A fuzzy raster map is a collection of cells whose values range between 0 and 1, specifying to which degree a particular attribute holds for the location represented by the cell. We showed that a brute-force buffering algorithm can be improved by using local propagation, which can be further improved by ordering the cells according to their membership grades.

The second part of the paper dealt with global buffering. The local buffer function $\beta(m)$ was replaced with a global buffer function $\psi(m, d)$, which gives us greater flexibility in updating membership grades. We introduced a brute-force algorithm for global buffering and showed several improvements of this algorithm, using similar techniques as in the first part of the paper.

The idea of using fuzzy set theory to handle imprecision in spatial reasoning is not new (Altmann 1994), and so it might look like a step backwards to consider buffer functions in a more rigid way than it is usually done. However, by focusing on first local and then global buffer functions, in particular their properties and capabilities, we are able to go beyond algorithms that perform brute-force buffering. This is of special interest if one is concerned about the computational complexity of buffering fuzzy maps.

There are a number of issues that we have not addressed in this paper. For example, we did not discuss how to choose a buffer function that, on the one hand, satisfies the required criteria for a local or global buffer function and, on the other hand, computes adequate membership grades. In general, determining adequate membership grades for a given fuzzy raster map is a problem. However, there are experiments showing that fuzzy membership grades are quite robust, which means that it is not necessary to have precise estimations of these grades (Bloch 2000). The explanation given for this observation is twofold: first, fuzzy membership grades are used to describe imprecise information and therefore do not have to be precise, and second, each individual fuzzy membership grade plays only a minor role in the whole reasoning process, as it is usually combined with several other membership grades. However, the ranking of membership grades must be preserved, which is in accordance with our findings in the context of spatial persistence (Guesgen & Hertzberg 1996).

## Acknowledgements

This research was partly supported by the University of Auckland Research Committee through various grants (most recently through grant number XXX/9343/3414100). We are grateful for the comments made by the two anonymous referees of this paper.

## References

Allen, J. 1983. Maintaining knowledge about temporal intervals. *Communications of the ACM* 26:832–843.

Altmann, D. 1994. Fuzzy set-theoretic approaches for handling imprecision in spatial analysis. *International Journal of Geographical Information Systems* 8(3):271–289.

Bloch, I. 2000. Spatial representation of spatial relationship knowledge. In *Proc. KR-00*, 247–258.

Brimicombe, A. 1997. A universal translator of linguistic hedges for the handling of uncertainty and fitness-of-use in GIS. In Kemp, Z., ed., *Innovations in GIS 4*. London, England: Taylor and Francis. 115–126.

Clementini, E.; Di Felice, P.; and Hernández, D. 1997. Qualitative representation of positional information. *Artificial Intelligence* 95(2):317–356.

Coppock, J., and Rhind, D. 1991. The history of GIS. In Maguire, D.; Goodchild, M.; and Rhind, D., eds., *Geographical Information Systems: Principles and Applications*. Essex, England: Longman Scientific & Technical. 21–43.

Driankov, D.; Hellendoorn, H.; and Reinfrank, M. 1996. *An Introduction to Fuzzy Control*. Berlin, Germany: Springer, 2nd edition.

Egenhofer, M., and Golledge, R. 1997. *Spatial and Temporal Reasoning in Geographic Information Systems*. New York: Oxford University Press.

Frank, A. 1992. Spatial concepts, geometric data models, and geometric data structures. *Computers and Geosciences* 18:409–417.

Frank, A. 1994. Qualitative temporal reasoning in GIS-ordered time scales. In *Proc. SDH-94*, 410–431.

Frank, A. 1996. Qualitative spatial reasoning: Cardinal directions as an example. *International Journal of Geographical Information Systems* 10(3):269–290.

Freksa, C. 1990. Qualitative spatial reasoning. In *Proc. Workshop RAUM*, 21–36.

Goodchild, M. 1992. Geographic data modeling. *Computers and Geosciences* 18:401–408.

Guesgen, H., and Albrecht, J. 1998. Qualitative spatial reasoning with a fuzzy distance operator. In *Proc. SDH-98*.

Guesgen, H., and Hertzberg, J. 1996. Spatial persistence. *Applied Intelligence (Special Issue on Spatial and Temporal Reasoning)* 6:11–28.

Guesgen, H., and Histed, J. 1996. Towards qualitative spatial reasoning in geographic information systems. In *Proc. AAAI-96 Workshop on Spatial and Temporal Reasoning*, 39–46.

Guesgen, H. 1989. Spatial reasoning based on Allen's temporal logic. Technical Report TR-89-049, ICSI, Berkeley, California.

Gupta, A.; Weymouth, T.; and Jain, R. 1991. An extended object-oriented data model for large image bases. In Günther, O., and Schek, H.-J., eds., *Advances in Spatial Databases*, Lecture Notes in Computer Science 525. Berlin, Germany: Springer. 45–61.

Hernández, D. 1991. Relative representation of spatial knowledge: The 2-D case. In Mark, D., and Frank, A., eds., *Cognitive and Linguistic Aspects of Geographic Space*. Dordrecht, The Netherlands: Kluwer. 373–385.

Herring, J. 1991. The mathematical modeling of spatial and non-spatial information in geographic information systems. In Mark, D., and Frank, A., eds., *Cognitive and Linguistic Aspects of Geographic Space*. Dordrecht, The Netherlands: Kluwer. 313–350.

Herring, J. 1992. TIGRIS: A data model for an object-oriented geographic information system. *Computers and Geosciences* 18:443–452.

Molenaar, M. 1996. The extensional uncertainty of spatial objects. In *Proc. SDH-96*, 9B.1–13.

Mukerjee, A., and Joe, G. 1990. A qualitative model for space. In *Proc. AAAI-90*, 721–727.

Peuquet, D. 1994. It's about time: A conceptual framework for the representation of temporal dynamics in geographic information systems. *Annals of the Association of American Geographers* 84(3):441–461.

Plewe, B. 1997. A representation-oriented taxonomy of gradation. In Hirtle, S., and Frank, A., eds., *Spatial Information Theory: A Theoretical Basis for GIS*, Lecture Notes in Computer Science 1329. New York: Springer. 121–136.

Raper, J., and Maguire, D. 1992. Design models and functionality in GIS. *Computers and Geosciences* 18:387–394.

Sinton, E. 1978. The inherent structure of information as a constraint to analysis: Mapped thematic data as a case study. In Dutton, G., ed., *Harvard Papers on Geographic Information Systems*, volume 6. Reading, Massachusetts: Addison-Wesley.

Usery, L. 1996. A conceptual framework and fuzzy set implementation for geographic features. In Burrough, P., and Frank, A., eds., *Geographic Objects with Indeterminate Boundaries*, GISDATA Series No. 2. London, England: Taylor and Francis. 71–85.

Zadeh, L. 1965. Fuzzy sets. *Information and Control* 8:338–353.